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Broadcasting on [0, L]

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Abstract

We study the problem of broadcasting in a system where n nodes are placed on a line of length L, independently uniformly distributed. We assume that every node is equipped with a transmitter whose radius of transmission is 1. We further assume that simultaneous broadcasts by neighbouring nodes results in garbled messages. The system is synchronous and time is slotted with nodes transmitting only during a slot.

We present an algorithm for broadcasting and compute the *expected* number of *time steps* required for it to complete. The algorithm is shown to be *optimal*. We also present and analyze an algorithm, to be executed by every node, that identifies the subset of nodes to serve as transmitters for a broadcast originating at the origin. We assume that the nodes are initially unaware of the topology of the system.

1. Introduction

Broadcasting is a problem related to information dissemination in communication networks. A node has a piece of information which needs to be transmitted to all the other nodes. Communication between a pair of nodes has traditionally been modelled as a *telephone call* during which the two nodes exchange all the information each of them has collected thus far.

Broadcasting algorithms have been studied for a variety of system models; see [7] for a comprehensive survey. In the basic system model it is assumed that any node may call any other node. If we represent each *possible* call by an edge in a graph whose n vertices represent the n nodes in the system, then the basic model assumes a *complete* graph. Another assumption is that a node may be engaged in only one call at a time. Given these constraints it is of interest to determine the *shortest sequence of calls* and the *minimum number of time steps* required to complete broadcasting (note that several calls may be made simultaneously). Several graph-based generalizations of the basic

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model have been studied, e.g., gossiping and broadcasting on hypergraphs [8], grid graphs [6] and trees [12].

In this paper we depart significantly from the graph-based models. We assume that nodes are equipped with radio transmitters. Thus when a node transmits, *all* nodes within range hear the message (barring situations where simultaneous transmissions lead to garbled messages). This model is thus very different from models where pairs of nodes communicate via telephone calls. It is noteworthy that our model is relevant to mobile radio networks where broadcasting is used to disseminate routing information.

In this paper we focus our attention on the one-dimensional case where all the nodes are placed on a line. We assume that the system consists of n nodes arranged uniformly *randomly* on a *line* of length L. Furthermore, we assume that every node is equipped with a radio transmitter with a transmission radius 1. A message transmitted by a node is received simultaneously by all the nodes within range. If two transmissions reach a node simultaneously then we assume that neither transmission is correctly received by that node (in other words the transmissions have *collided*).

The problem of broadcasting in *packet radio networks* has been addressed by several authors. Some authors have used a graph-based model to represent the system [5, 9] and present efficient algorithms to broadcast, see, e.g., [1–3]. Other authors represent the system as a collection of nodes located randomly on a line or an infinite two-dimensional plane and study issues related to connectivity, routing and performance evaluation [4].

Our primary focus in this paper is to determine the *expected minimum time* required to broadcast when nodes are placed uniformly randomly on the interval $[0, L]$. We present an algorithm for broadcasting and show that it is optimal.

Our broadcasting algorithm (presented in Section 3) assumes that every node knows the identity of its furthest neighbour, i.e., the most distant node within transmission range. In Section 4 we present an algorithm that enables nodes to identify these neighbours in random system configurations; here we assume that the nodes initially have no knowledge of the system configuration.

1.1. Main results and outline

This paper is organized as follows. In the next section we provide formal definitions for our system model and define the metrics of interest. Section 3 deals with the problem of broadcasting on $[0, L]$. The problem of identifying transmitters is studied in Section 4. Section 5 presents a summary of the main results and Section 6 presents our conclusions and describes open problems.

In the section on broadcasting, we first present an optimal algorithm to broadcast and then devote the rest of the section to an analysis of the complexity of this algorithm. The complexity to broadcast is a function of both n, the number of nodes, and L, the length of the line. We present expressions for the expected number of time steps required to broadcast when L is a constant as well as when L is unrestricted.

Finally, we study the asymptotic complexity as $n \to \infty$ for the cases when $L \sim n^{\alpha}$ and when $L \sim \alpha n$. We show that in most of the cases the number of steps to broadcast is L except when $L \sim \alpha n$, $\alpha > 3/4$. In this case the number of steps required is $3L/4\alpha$.

In Section 4 we present an algorithm that identifies a subset of nodes to serve as "relays" (called boundary nodes) for a broadcast originating at the origin. We conjecture that this algorithm is optimal.

2. The model

Let $n + 1$ denote the number of nodes in the system. We assume that the nodes are placed *randomly* and independently on a *line* of length L. Every node has a transmitter with a transmission radius of 1. Thus nodes i and j are within range of each other if the distance between them is less than 1.

Let us call any particular arrangement of the n nodes on the line a *configuration*. A configuration is *connected* if every node is reachable from every other via a series of transmissions. Clearly a large fraction of possible configurations will not be connected. In this paper, however, we are only interested in connected configurations.

A final aspect of our model deals with the problem of simultaneous transmissions. In this paper we take into consideration two types of *collisions*. The first kind of collision occurs when two or more nodes that are within transmission range of each other (i.e., the distance between them is less than 1) transmit simultaneously, see Fig. 1(a). All the nodes hear noise. The second kind of collision is possible when three nodes 1, 2 and 3 are placed in such a way that node 3 is within range of both 1 and 2. If 1 and 2 transmit simultaneously, node 3 hears only noise, however, neither 1 nor 2 is aware that 3 heard noise. See Fig. 1(b).

For the system model presented above we are interested in estimating the expected value of the *minimum number of time steps* required to broadcast in connected configurations. We assume that time is slotted and the system is synchronous. Furthermore, we assume that a transmission lasts for exactly one slot.

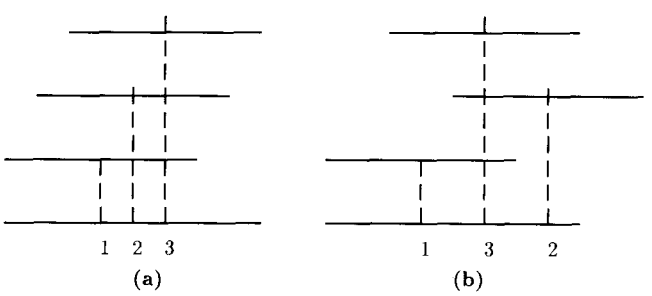

Fig. 1. Node 3 hears noise if 1 and 2 transmit simultaneously.

3. Broadcasting

Broadcasting refers to the problem of efficiently disseminating information from one node to all the other nodes in the system. In the model considered in this paper, the number of time steps required to broadcast depends upon the specific configuration of the nodes. If all the nodes are grouped closely together within transmission radius of each other then the time to broadcast is 1 while if the nodes are spread out we would require more time steps. Therefore in order to determine the *expected* value of the minimum number of time steps required to broadcast, we need to categorize the set of connected configurations of n nodes on the line.

Let us assume that all the nodes are placed on a line length L. Let the node that initiates the broadcast, node x_0, be at position 0. The remaining nodes are located in the region $(0, L]$ (the case where the initiating node, node x_0, has nodes to both sides of it is a simple generalization). Let us assume that the node most distant from node x_0 is located at position T. Then it is clear that we would require at least $\lceil T \rceil$ time steps to broadcast (because each transmission can cover a distance of 1). The worst-case number of time steps required is, however, almost double!

Lemma 3.1. *In the worst case the number of time steps required to broadcast from node* x_0 *is* $2\lfloor T \rfloor + 1$.

Proof. In order to prove the lemma, we first provide a construction that requires $2\lfloor T \rfloor + 1$ time steps and then show that it is not possible to do any worse.

Let $T = K + \delta$ where K is an integer and $0 < \delta < 1$. Let x_j and $y_j, j = 0, 1, \ldots, \lfloor T \rfloor$, represent the positions of $2\lceil T \rceil$ nodes with the following properties:

$$0 < y_j - x_j < \varepsilon, \quad \text{and } x_0 = 0, \quad y_{\lfloor T \rfloor} = T,$$

$$x_{i+1} = x_i + 1 + \varepsilon/2,$$

where $\lceil T \rceil \varepsilon < \delta/2$. All the remaining $n - 2\lceil T \rceil$ nodes are placed randomly in the intervals $(y_{j-1} + 1, y_j)$ for $j > 0$ and the interval (x_0, y_0).

In order to broadcast from the node at position 0 (i.e., x_0) the sequence of transmissions required is $x_0, y_0, x_1, y_1, \ldots, x_{\lfloor T \rfloor}$. This is easy to see because node x_j is out of range of x_{j-1} but within range of y_{j-1}. On the other hand, node y_j is out of range of y_{j-1}. Therefore, to get a message from x_{j-1} to y_j, the sequence of transmissions needed is x_{j-1}, y_{j-1}, x_j. A simple inductive argument will convince the reader that, in general, to get from node x_0 to y_k, $2k + 1$ transmissions are required. Therefore, in the line we require $2\lfloor T \rfloor + 1$ transmissions; see Fig. 2.

In order to see that we cannot do any worse, it is sufficient to point out that with two successive transmissions, all the nodes within a distance $1 + \varepsilon$ (for some $1 > \varepsilon > 0$) of the first node can be reached. Thus, if the most distant node is located at $T = K + \delta$ distance away from the originating node then at most $2K$ transmissions

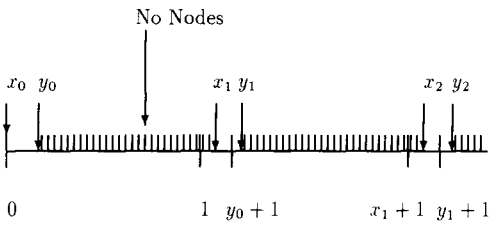

Fig. 2. Worst-case scenerio.

are required to inform a node that is less than δ distance away from the final node and one more transmission is required to inform the final node. □

For arbitrary connected configurations on a line it is of interest to develop an algorithm that permits broadcasting in a minimum number of time steps. We present an algorithm below which is clearly *optimal* in the sense that for any particular configuration it requires the fewest number of time steps.

Algorithm 1. Let the node that begins the broadcast be represented as x_0. Node x_0 broadcasts at time step 1. Identify nodes $x_1, x_2, \ldots, x_k$ with the property that x_i is the most distant node to the right of x_{i-1} still within transmission range of x_{i-1}, i.e., $x_{i-1} < x_i < x_{i-1} + 1$. Therefore, x_k is the *last* node to the right of x_0.

The sequence of transmissions is now $x_0, x_1, \ldots, x_{k-1}$. In time step 1, node x_0 transmits. In time step 2, the node x_1 transmits, and so on. At the end of k time steps (when node x_{k-1} transmits) all the nodes have received the message. Thus the number of time steps required is exactly k. □

The rest of this section addresses the issue of determining the *expected* time complexity of broadcasting for Algorithm 1. Section 4 deals with the *algorithmic* problem of identifying the nodes x_i (from Algorithm 1 above) in a *real* system where the nodes are initially unaware of the configuration. For this we present a protocol that is executed by each node with the property that at termination, all the transmitters (the nodes x_i) have been identified.

3.1. Complexity of broadcasting

We are interested in determining the expected value of the number of time steps required to broadcast from node x_0 in random configurations. Let p_k denote the probability that exactly k steps are required. Then,

$$E[\text{time steps}] = \sum_{k=1}^{m} k p_k, \tag{1}$$

where, $m = \min\{2\lfloor L \rfloor + 1, n\}$ and $\sum_{k=1}^{m} p_k = 1$. If the length of the line is L then by Lemma 3.1 above, in the worst case $2\lfloor L \rfloor + 1$ time steps are required to broadcast. On the other hand, if the number of nodes available is small, $n < 2\lfloor L \rfloor + 1$, then in the worst case exactly n transmissions will be required to broadcast (node x_0 is at location 0 and every succeding node is located a distance $1 - \varepsilon$, $\varepsilon \ll 1$ to the right of the previous node).

In order to compute the value of p_k we break our analysis into two cases. In the first case, we assume the length of the line, L is *unrestricted*, i.e., the only constraint we impose will be one of connectivity. In this case, therefore, the number of boundary nodes varies from 1 to n. In the second case we restrict the length of the line to be exactly $L \ll n$ and therefore the number of boundary nodes varies between 1 and $2\lfloor L \rfloor + 1$.

3.1.1. Unrestricted length case

Let us construct the set of configurations for which k transmissions are required. We first place nodes $x_1, x_2, \ldots, x_k$ to the right of node x_0 (which is located at position 0) such that node x_i is uniformly distributed between $x_{i-2} + 1$ and $x_{i-1} + 1$. Node x_1 is placed in the region between 0 and 1. These nodes are called the *boundary nodes*. We have thus placed k nodes and still need to place the remaining $n - k$ nodes. (We are assuming that there are $n + 1$ nodes in the system including the node at 0.)

In order to ensure that exactly k time steps are required to broadcast in this set of configurations, there are certain regions where no nodes may be placed. The union of these regions is called the *forbidden region*. In Fig. 3 the forbidden region is indicated by the shaded area. We can therefore write,

$$(x_i, x_{i-1} + 1), \quad 1 < i < k.$$

The total *length* of the forbidden region is therefore

$$(1 - x_1) + (x_1 + 1 - x_2) + \cdots + (x_{k-2} + 1 - x_{k-1}).$$

We can now determine the length of the *allowable region* where the $n - k$ nodes may be placed. The length of this region is

$$R = x_1 + (x_2 - (x_0 + 1)) + \cdots + (x_k - (x_{k-2} + 1)) = x_k + x_{k-1} - (k - 1). \quad (2)$$

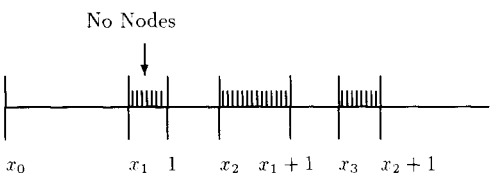

Fig. 3. Forbidden region.

Define

$$p_{k,n}(x) \triangleq \Pr[k \text{ boundary nodes} \mid \text{connected}],$$

where n is the number of nodes and $(0, x)$ is the region in which the nodes are placed. For the unrestricted length case note that we are interested in determining $p_{k,n}(x)$ *for* $x > k$ because all connected configurations with k boundary nodes will lie within the region $(0, k)$. Let us denote this probability for the unrestricted case as $p_{k,n}(\infty)$.

Combining the ideas presented above, we can now write the expression for the probability of exactly k boundary nodes as

$$\Pr[k \text{ boundary nodes} \mid \text{connected}] = \frac{\Pr[k \text{ boundary nodes and connected}]}{\Pr[\text{connected}]}. \tag{3}$$

The probability of a connected configuration is the ratio of the *volume*[1] of all connected configurations on $(0, x)$ to the volume of all configurations on the same interval. Therefore,

$$\Pr[\text{connected}] = \frac{n!}{x^n} \int_0^1 dz_1 \int_{z_1}^{z_1+1} dz_2 \cdots \int_{z_{n-1}}^{z_{n-1}+1} dz_n = \frac{n!}{x^n}.$$

The $n!$ arises because we consider all permutations of the points $x_1, x_2, \ldots, x_n$. Also it is noteworthy that $n!/x^n < 1$ since the n nodes are placed between $(0, x)$.

Similarly, the probability in the numerator of Eq. (3) is the ratio of the volume of all connected configurations with exactly k boundary nodes to the volume of all configurations. We construct connected configurations with k boundary nodes as described earlier. First select k nodes out of n to serve as boundary nodes and then place the remaining $n - k$ nodes uniformly in the *allowable region.*

Using Eq. (3) we can now write $p_{k,n}(\infty)$, after simplification as,

$$p_{k,n}(\infty) = \frac{1}{(n-k)!} \int_0^1 dx_1 \int_1^{x_1+1} dx_2 \int_{x_1+1}^{x_2+1} dx_3 \cdots \int_{x_{k-2}+1}^{x_{k-1}+1} R^{n-k}\, dx_k. \tag{4}$$

Observe that the integrand in the expression above, R, depends only upon x_k and x_{k-1}. We use this fact to construct an *upperbound* for $p_{k,n}(\infty)$. Let us place the boundary nodes such that the boundary node x_j is uniformly distributed in the interval $(x_{j-1}, x_{j-1} + 1)$. The remaining nodes are placed in the *allowable region* as we

[1] Geometrically, the *volume* of all the possible configurations on $(0, x)$ is the volume of the n-dimensional cube with each side of length x, which is x^n. Observe that each point in this cube represents a different configuration. The volume of the connected configurations is the volume of a n-dimensional polyhedron within the cube and defined by the limits of the definite integral.

have previously discussed. We can now write

$$p_{k,n}(\infty) \leqslant \frac{1}{(n-k)!} \int_0^1 \mathrm{d}x_1 \int_{x_1}^{x_1+1} \mathrm{d}x_2 \cdots \int_{x_{k-1}}^{x_{k-1}+1} R^{n-k} \chi\left\{x_{k-1} > \left\lfloor \frac{k-1}{2} \right\rfloor, x_k > \left\lfloor \frac{k}{2} \right\rfloor\right\} \mathrm{d}x_k, \tag{5}$$

where χ is a characteristic function that takes a value 0 if the condition is not satisfied and 1 if it is. We need the characteristic function in the approximation because the limits of the integral no longer guarantee that the integrand will be positive and this is because it is possible for *all* the boundary nodes to lie within (0, 1). By requiring that $x_{k-1} > (k-1)/2$, we ensure that no more than a constant number of boundary nodes lie within transmission range of one another. See Fig. 4.

In this new form, x_k and x_{k-1} are uniform i.i.d. random variables. Therefore, the above integral can be simplified to

$$p_{k,n}(\infty) \leqslant \frac{1}{(n-k)!} \iint_{\mathscr{H}} R^{n-k} f_{k-1}(x_{k-1})\, \mathrm{d}x_k\, \mathrm{d}x_{k-1}, \tag{6}$$

where the integral is carried out over the region $\mathscr{H}$ as shown in Fig. 4. $f_{k-1}(x_{k-1})$ is the density of the $(k-1)$-fold convolution of uniform i.i.d. random variables. In Section 3.1.3 we derive several asymptotic properties of $p_{k,n}$ and E[time steps]. In the next section, we derive an expression for $p_{k,n}(L)$ for the restricted length case.

3.1.2. Restricted length case

Let us now consider the case where the length of the interval is fixed. Therefore, all the nodes are placed uniformly between 0 and L. As above, we are interested in determining the probability of k boundary nodes given that the configurations are connected. There are three cases to consider depending on the value of k, the number of boundary nodes. The cases are $k < \lfloor L \rfloor$, $\lfloor L \rfloor \leqslant k \leqslant 2\lfloor L \rfloor + 1$ and $k > 2\lfloor L \rfloor + 1$. It is easy to see that

$$p_{k,n}(L) = \begin{cases} \dfrac{p_{k,n}(\infty)}{\Pi_n(L)}, & k < \lfloor L \rfloor, \\ 0, & k > 2\lfloor L \rfloor + 1, \end{cases} \tag{7}$$

where $\Pi_n(L)$ is the probability that the sum of n i.i.d. uniform random variables is less than L, i.e., $\Pi_n(L) = \int_0^L f_n(x)\, \mathrm{d}x$.

In the expression above when $k < \lfloor L \rfloor$, the set of all connected configurations with k boundary nodes is a proper subset of the set $(0, L]^n$ and therefore the probability is just as in the unrestricted case. By Lemma 3.1, k cannot be larger than $2\lfloor L \rfloor + 1$ and the probability for this case is therefore 0.

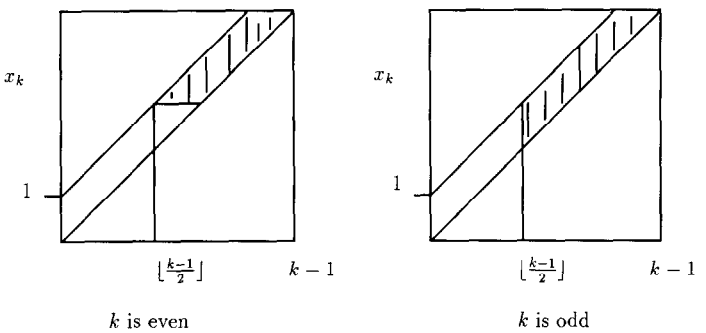

Fig. 4. The region $\mathscr{H}$.

For the case that $\lfloor L \rfloor \leqslant k \leqslant 2\lfloor L \rfloor + 1$ we use an approach very similar to the one used previously to determine $p_{k,n}(\infty)$. We select k nodes out of n and identify them as the boundary nodes. These nodes are placed such that $x_k < L$. The remaining nodes are placed in the *allowable* region, R, with a uniform distribution.

It is easy to see that the length of the allowable region is the same as in the unrestricted case. Therefore, we can write, as in Eq. (4),

$$p_{k,n}(L) = \frac{1}{(n-k)!} \int_0^1 \mathrm{d}x_1 \int_1^{x_1+1} \mathrm{d}x_2 \cdots \int_{x_{k-2}+1}^{x_{k-1}+1} R^{n-k} \chi\{x_k < L\} \, \mathrm{d}x_k, \tag{8}$$

where χ is a characteristic function that takes the value 0 if $x_k > L$ and 1 otherwise.

As in the unrestricted case, we provide an upperbound for this integral using an identical construction. Thus,

$$p_{k,n}(x) \leqslant \frac{1}{(n-k)!} \iint_{\mathscr{H}'} R^{n-k} f_{k-1}(x_{k-1}) \mathrm{d}x_k \mathrm{d}x_{k-1}, \tag{9}$$

where the integral is carried out over the region $\mathscr{H}'$. There are two cases we must consider; if $k-1 < x \leqslant k$ then the region of interest if $\mathscr{H}'_a$ in Fig. 5; if $x \leqslant k-1$ then the region of interest is $\mathscr{H}'_b$.

3.1.3. Asymptotic results

In this section we study asymptotic properties of the expected number of time steps required to broadcast for large values of n. Let N_n denote the number of time steps required to broadcast (or equivalently, the number of boundary nodes). Then it is intuitively clear that when $n \to \infty$, $E[N_n] = L$ if L is a constant. If $L = L(n)$ however, it is not obvious how $E[N_n]$ would behave. In this section we characterize the behaviour of $E[N_n]$ for the case when $L(n) \sim n^\alpha$, $0 < \alpha < 1$ and when $L(n) \sim \alpha n$, $\alpha > 0$. First however, we prove an asymptotic result concerning $p_{k,n}$.

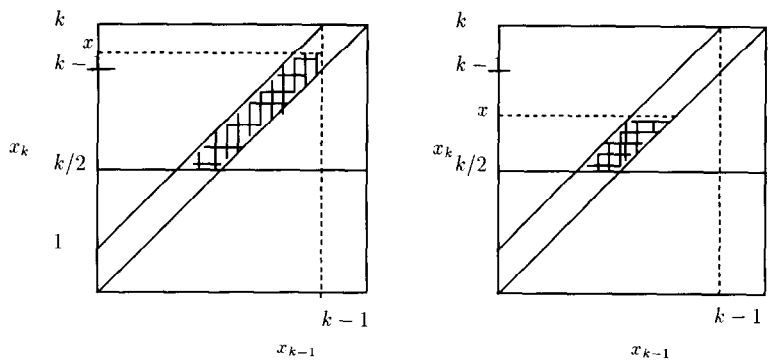

Fig. 5. The regions $\mathscr{H}'_a$ and $\mathscr{H}'_b$.

Theorem 3.2 and Lemma 3.3 below elucidate the behaviour of $p_{k,n}(L)$ for large n when $L(n) \sim n^\alpha$, $\alpha \leqslant 1/2$. We first show that $p_{k,n}(L) = 0$ if $k \neq L$ and 1 if $k = L$. In Lemma 3.3 we show how this asymptote is achieved.

Theorem 3.2. *Let $L(n)$ be a positive integer valued function such that $L(n)/n^\alpha \to c > 0$ as $n \to \infty$ for some $0 \leqslant \alpha < 1/2$. Then*

$$\lim_{n\to\infty} p_{L(n),n}(L) = 1.$$

Proof. In order to prove this theorem, we will construct a special set of configurations that have L boundary nodes. We will then show that a random connected configuration lies in this set with a probability approaching 1.

Let $0 < \delta_n < 1$, and $\delta_n \ln(n) \to b > 0$, $n \to \infty$. Consider the L intervals, $(1 - \varepsilon_n, 1)$, $(2 - 2\varepsilon_n, 2 - \varepsilon_n), (3 - 3\varepsilon_n, 3 - 2\varepsilon_n), \ldots, (L - L\varepsilon_n, L - (L - 1)\varepsilon_n)$, $0 < \varepsilon_n < \delta_n/L$. Let G_n denote the set of configurations in $(0, L]^n$ which have at least one node in each of the intervals listed above. Let $B_n \subseteq G_n$ be the set of configurations with no nodes in the interval $(L(n) - (L(n) - 1)\varepsilon_n, L(n)]$. From the definition of G_n and B_n it is clear that every configuration in G_n has at least L boundary nodes and that every configuration in B_n has at most L boundary nodes. Moreover G_n (and hence B_n) is a subset of the set of connected configurations in $(0, L]$, $C_L(n)$. Since $\delta_n \sim [\ln n]^{-1}$, it is easy to see that $\Pr(G_n) - \Pr(B_n) \to 0$ as $n \to \infty$.

$$\Pr(G_n) \geqslant 1 - L\left(1 - \frac{\varepsilon_n}{L}\right)^n \geqslant 1 - L\left(1 - \frac{\delta_n}{L^2}\right)^n.$$

Since $\delta_n \sim [\ln n]^{-1}$ and $L \sim n^\alpha$, $\alpha < 1/2$, it easily follows that $L(1 - \delta_n/L^2)^n \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

The theorem above talks about the behaviour of the probabilities in the limiting case only. It is interesting to see how this limit is achieved. Lemma 3.3 gives us an indication of how the probabilities behave for large n.

Lemma 3.3. *Let $L(n)$ be as in Theorem 3.2. Let $\Pi_n(L) = \int_0^L f_n(x)\,dx$ be the probability that the sum of n i.i.d. $[0, 1]$ uniform random variables is less than or equal to L. We have the following bounds.*

$$p_{k,n}(L) \leqslant \begin{cases} \dfrac{1}{\Pi_n(L)}\dfrac{1}{(n-k)!}\displaystyle\int_0^1 dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_{k-1}}^{x_{k-1}+1} dx_k\, k^{n-k}, & \text{if } k \leqslant L, \\ \dfrac{1}{\Pi_n(L)}\dfrac{1}{(n-k)!}\displaystyle\int_0^1 dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_{k-1}}^{x_{k-1}+1} dx_k\, (2L-k)^{n-k}, & \text{if } L < k \leqslant 2L. \end{cases}$$

Proof. The bounds are obtained from Eq. (9) for $p_{k,n}(L)$. We bound R (the allowable region; see Fig. 3 and Eq. (2)) by the maximum possible allowed region. Since the integrand thus obtained is positive we replace the integrals by integrals over a larger region giving us an upperbound.

We now show that the bounds for $k \neq L$ converge to zero, while the bound for $k = L$ converges to 1 as $n \to \infty$. Since $B_n \subseteq C_L(n)$ (recall that $C_L(n)$ is the set of connected configurations on $(0, L]$) and $\Pr(B_n) \to 1$ as $n \to \infty$ we have that $\Pr(C_L(n)) \to 1$ as $n \to \infty$. Thus

$$\Pr(C_L(n)) = \frac{n!\,\Pi_n(L)}{L^n} \to 1, \quad \text{as } n \to \infty.$$

Consider the bound when $k < L$.

$$\begin{aligned} &\frac{1}{\Pi_n(L)}\frac{1}{(n-k)!}\int_0^1 dx_1 \cdots \int_{x_{k-1}}^{x_{k-1}+1} dx_k\, k^{n-k} \\ &\quad = \frac{1}{\Pi_n(L)}\frac{L^{n-k}}{(n-k)!}\frac{k^{n-k}}{L^{n-k}} \\ &\quad \leqslant \frac{\Pi_{n-k}(L)}{\Pi_n(L)}\frac{L^{n-k}}{\Pi_{n-k}(L)(n-k)!}\left(\frac{L-1}{L}\right)^{n-k}. \end{aligned}$$

Let

$$\frac{\Pi_{n-k}(L)}{\Pi_n(L)}\frac{L^{n-k}}{\Pi_{n-k}(L)(n-k)!} = D_{n,k}.$$

Since $\Pi_{n-k}(L)/\Pi_n(L) \to 1$ as $n \to \infty$, $D_{n,k} \to 1$ as $n \to \infty$ for $0 < k \leqslant 2L$. Thus, $D_{n,k}(1 - 1/L)^{n-k} \to 0$ as $n \to \infty$ for all $0 < k < L$. Note that when $k = L$, the limit equals one. A similar computation shows that the bound limits to zero when $k > L$. □

The rest of this section is devoted to a characterization of N_n for various $L(n)$. We first show that $E[N_n]$ converges to L when $L(n) \sim n^\alpha$, $\alpha \leqslant 1/2$. Next we show that N_n converges to L in probability when $1/2 < \alpha < 1$. The case when $L(n) \sim \alpha n$ is discussed at the end of this section.

Corollary 3.4. *Let $L(n)$ be as in Theorem 3.2. Then*

$$\lim_{n\to\infty} \frac{E[N_n]}{L} = 1.$$

Proof.

$$E[N_n] = Lp_{L,n}(L) + \sum_{k=0}^{L-1} kp_{k,n}(L) + \sum_{k=L+1}^{2L-1} kp_{k,n}(L).$$

The first sum in the equation above may be written as

$$\begin{aligned}\sum_{k=0}^{L-1} kp_{k,L}(L) &\leqslant C_n \sum_{k=0}^{L-1} k\left(\frac{L-1}{L}\right)^{n-k} \\ &\leqslant C_n\left(\frac{L-1}{L}\right)^{n-k} L^2 \\ &\leqslant C_n\left(\frac{L-1}{L}\right)^{n-L} L^2,\end{aligned}$$

where $C_n \to 1$ as $n \to \infty$. It is easy to show that $(1 - 1/L)^{n-L}L^2 \to 0$ as $n \to \infty$.

Similarly we prove that $\sum_{k=L+1}^{2L-1} kp_{k,n}(L) \to 0$ as $n \to \infty$. This completes the proof of the corollary. □

The above result proved *absolute convergence* of the expected number of boundary nodes to L for the case that $L \sim n^\alpha$ for $\alpha < 1/2$. Theorem 3.5 proves *convergence in probability* of the number of boundary nodes to L for the more general case when $0 < \alpha < 1$.

Theorem 3.5. $\forall \alpha$, $0 < \alpha < 1$, *if* $\lim_{n\to\infty} L(n)/n^\alpha = 1$ *then,*

$$\forall \delta > 0, \quad \lim_{n\to\infty} P_n\left(\left|\frac{N_n}{L(n)} - 1\right| > \delta\right) = 0,$$

where N_n is the number of boundary nodes.

Proof. Suppose $\lim_{n\to\infty} L(n)/n^\alpha = 1$ for some $0 < \alpha < 1$. Let $\varepsilon > 0$. Put

$$M_n = \sup\{K \in \mathcal{N} \mid K(1-\varepsilon) < L\},$$

where $\mathcal{N}$ is the set of natural numbers. Note that

$$\frac{L-1}{1-\varepsilon} < M_n < \frac{L}{1-\varepsilon}.$$

Let E_n be the event that there is at least one node in each of the intervals $(1-\varepsilon, 1]$, $(2-2\varepsilon, 2-\varepsilon], \ldots, (M_n(1-\varepsilon), M_n(1-\varepsilon)+\varepsilon]$.

$$P_n(E_n) \geqslant 1 - M_n\left(1 - \frac{\varepsilon}{L(n)}\right)^n.$$

Since $L(n) \sim n^\alpha$ and $M_n \sim L(n)$ it easily follows that $\lim_{n\to\infty} M_n(1-\varepsilon/L(n))^n = 0$. Thus,

$$\lim_{n\to\infty} P_n(E_n) = 1.$$

We now note that when E_n occurs $N_n \leqslant M_n + 1$ and $N_n \geqslant L - 1$. Therefore,

$$P_n\left(-\frac{1}{L(n)} \leqslant \frac{N_n}{L(n)} - 1 \leqslant \frac{\varepsilon}{1-\varepsilon} + \frac{1}{L(n)}\right) \geqslant P(E_n),$$

since $1/L(n) \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} P_n\left(\left|\frac{N_n}{L(n)} - 1\right| \leqslant \frac{2\varepsilon}{1-\varepsilon}\right) = 1,$$

which proves the theorem. □

Finally, we consider the case that $L(n) \sim \alpha n$ for α a constant. As earlier, let N_n denote the number of boundary nodes. Then we have the following two results.

Theorem 3.6 (*Law of large numbers and central limit theorem*).

(a) $\forall \varepsilon > 0$, $P_n\left(\left|\frac{N_n}{n} - 3/4\right| > \varepsilon\right) \to 0$, *as* $n \to \infty$,

(b) $\frac{1}{\sqrt{3n/16}}[N_n - E(N_n)] \to N(0, 1)$ *in distribution as* $n \to \infty$.

This theorem is proved in [10]. Intuitively, according to the theorem, the number of boundary nodes converges to $3n/4$ in probability!

Using Theorem 3.6 for N_n we can now extend our results about the relationship between N_n and $L(n)$ further. The first case is when $L(n) > n$ and $\lim_{n\to\infty} L(n)/n = \alpha$. We know from the above theorem that $N_n/n \xrightarrow{P} 3/4$ (the P above the arrow indicates convergence in probability as n goes to infinity), therefore, we have $N_n/L \xrightarrow{P} 3/4\alpha$.

Suppose $3/4n \leqslant L < n$ and let $\lim_{n\to\infty} L/n = \alpha$ then by the central limit theorem (CLT) for N_n we know that if we start with a system of length n and restrict the configurations to be between 0 and $L(n)$ then asymptotically the restricted set will

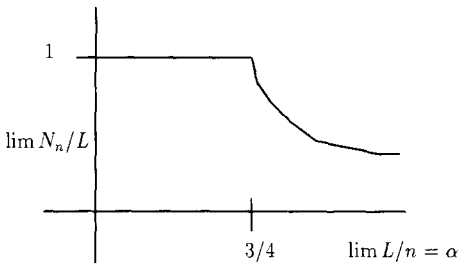

Fig. 6. Relationship between $L(n)$ and N_n.

Table 1
Asymptotic results for time to broadcast (N_n)

$L \sim n^{\alpha}$, $0 < \alpha < 1/2$	$\lim_{n\to\infty} E[N_n]/L = 1$	Absolute convergence
$L \sim n^{\alpha}$, $0 < \alpha < 1$	$\lim_{n\to\infty} P_n(\lvert N_n/L - 1\rvert > \varepsilon) = 0$	Convergence in probability
$L \sim n\alpha$, $\alpha \leqslant 3/4$	$\lim_{n\to\infty} P_n(\lvert N_n/L - 1\rvert > \varepsilon) = 0$	**Conjecture**
$L \sim n\alpha$, $\alpha > 3/4$	$\lim_{n\to\infty} P_n(\lvert N_n/L - 3/4\alpha\rvert > \varepsilon) = 0$	Convergence in probability

have a probability $\geqslant 1/2$. Therefore, $N_n/n \xrightarrow{P} 3/4$ holds for the restricted probability distributions. So we again have, $N_n/L \xrightarrow{P} 3/4\alpha$.

Finally, if $\alpha < 3/4$ we cannot use the CLT as before since the set of restricted configurations will asymptotically have a probability zero. We now have the following conjecture.

Conjecture 3.7. If $\lim_{n\to\infty} L(n)/n = \alpha$ then,

$$\frac{N_n}{L(n)} \xrightarrow{P} 1,$$

if $\alpha < 3/4$.

The relationship between N_n/L and L/n is best illustrated by Fig. 6. We summarize all the asymptotic results in Table 1.

4. Protocol to identify boundary nodes

We now focus our attention on the *algorithmic* problem of identifying the boundary nodes. This is necessary to avoid collisions during a broadcast. Thus, after x_0 initiates a broadcast, x_1 should be the only node to transmit in the next step. If other nodes between x_0 and x_1 transmit in this step, there will be collisions. The algorithm described below identifies all the boundary nodes in any connected configuration.

Assume that every node is aware of its *position* (i.e., its coordinate in $[0, L]$) in the configuration. At the start of the algorithm only the node at 0 is aware that it is a boundary node. At termination, all the nodes that are boundary nodes by our earlier definition are aware that they are the boundary nodes.

Algorithm 2.

1. The boundary node at position x_i transmits a START$(i+1, l_i, l_{i-1})$ message to begin looking for the boundary node x_{i+1}. l_i is the position of boundary node x_i and l_{i-1} is the position of boundary node x_{i-1}.

2. All the nodes to the *right* of x_i that hear its broadcast transmit in the next time step. If there are no nodes to the right of x_i, there is silence and node x_i terminates the algorithm by sending a TERMINATE(i) message that will be propagated succesively backwards by all the newly found boundary nodes.

3. If there is exactly one node to the right of boundary node x_i, it hears a successful transmission (i.e., no noise) and concludes that it is the boundary node x_{i+1}. This node then restarts the algorithm at step 1.

4. If there are more than one node to the right of x_i, the simultaneous transmissions in step 2 will cause all the nodes that transmitted to hear noise. Let $R = (r_0, r_1]$ denote the *region* where the boundary node x_{i+1} is to be found. Initially, $r_0 = l_{i-1} + 1$ and $r_1 = l_i + 1$. Note that the region $(l_i, l_{i-1} + 1]$ is part of the *forbidden region* and there will therefore be no nodes here.

(a) Let $0 < q < 1$ be a number. Let $\Delta = (r_1 - r_0) * q$. All nodes in the region $(r_1 - \Delta, r_1]$ transmit.[2]

(i) If there is only one transmission, then the node making it identifies itself as the boundary node x_{i+1}. GOTO step 1.

(ii) If there is more than one transmission then the nodes hear noise. Define $r_0 = r_1 - \Delta$ and GOTO step (a).

(iii) There is silence implying that there are no nodes in the region. GOTO step (b).

(b) Do a backward sequential search through regions of size Δ until we find a region with at least one node in it.

```
j = 1;
while (Silence) do
begin
j = j + 1;
All nodes in (r1 − j*Δ, r1 − (j − 1)*Δ] transmit;
end;
```

Eventually in some step $j = k$ one or more nodes transmit. If there is exactly one transmission, that node identifies itself as the boundary node x_{i+1}, GOTO step 1. If

[2] A simpler method would be to perform a binary search. First nodes in the region $[(r_0 + r_1)/2, r_1]$ transmit, if there is silence redefine r_1 to be $(r_0 + r_1)/2$, otherwise redefine r_0 to be $(r_0 + r_1)/2$. We need the extra complication with q in order to optimize the time required to find one boundary node.

there is noise, define $r_0 = r_1 - k * \Delta$ and $r_1 = r_0 + \Delta$ and GOTO step (a); the boundary node will be in this region.

For purposes of exposition, let us focus on finding the boundary node x_1 in the region $(0, 1]$. Let us assume that there are m nodes in this region. Node x_0 transmits a START$(1, 0, .)$ message in step 1 to begin the algorithm. In step 2, all the $m > 0$ nodes transmit. If $m > 1$ then all the nodes will hear noise. Let us assume this to be the case. Nodes in the region $(1 - q, q]$ transmit (step 4(a)) next. If there is noise, nodes in the region $(1 - q^2, 1]$ transmit in the next step, and so on. Assume that until the sth time step when nodes in $(1 - q^s, 1]$ transmit, there is always a noisy transmission. In the sth time step assume that there is silence (i.e., there are no nodes in this region). By step 4(b), nodes in $(1 - 2q^s, 1 - q^s]$ transmit next; if there is silence, then nodes in $(1 - 3q^s, 1 - 2q^s]$ transmit and so on until eventually there is at least one transmission in the region $(1 - kq^s, 1 - (k - 1)q^s]$. At this point we have either located the boundary node x_1 (one transmission) or the algorithm returns to step 4(a) (noisy transmission) with $R = (1 - kq^s, 1 - (k - 1)q^s]$.

Note that step 4(b) represents a *sequential search* through regions of size Δ while step 4(a) represents a *logarithmic* search. Let us call each execution of step 4(a) a *phase*. During one phase, there may be more than one execution of step 4(b). The total time complexity of finding *one* boundary node is the total number of time steps required.

4.1. Complexity of finding one boundary node

If there is no information available about the distribution of the nodes in $(0, 1]$, then if $q = 1/2$ we require $\log_2(1/D)$ time steps to find one boundary node (D is the minimum separation between two nodes). If, on the other hand, we are given that the distribution of the nodes in $[0, 1]$ is uniform then, as we show below, we can find the boundary nodes in fewer time steps.

Our first result in Theorem 4.1 bounds the expected number of executions of step 4(a) in Algorithm 2. (Recall that each such execution is called a *phase*.) The subsequent results compute the number of executions of step 4(b) for each execution of step 4(a). Using these results, we then conjecture that the "best" value for q is $1/\log_2 m$, where m is the number of nodes in any region of length 1.

Theorem 4.1 (*Expected number of phases*). *Let $X_1, X_2, \ldots, X_m$ be i.i.d. uniform, $[0, 1]$ r.v.'s. Let N represent the number of phases required to find the largest of the X_i using Algorithm 2 above. Then,*

$$E[N] < B\frac{\ln m}{\ln(1/q)}$$

for $m \to \infty$. B is a constant and q is the fraction used in Algorithm 2.

Proof. Let $(X_1, X_2, \ldots, X_m)$ be i.i.d uniform random variables on $[0,1]$. Let $Y = \max_{1 \leqslant i \leqslant m} X_i$, $W = \max_{1 \leqslant i \leqslant m}\{X_i : X_i < Y\}$. Let $f(y,w)$ be the joint probability density of Y and W.

$$f(y,w) = \begin{cases} m(m-1)w^{m-2}, & \text{if } 0 < w < 1,\ w < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note: If we assume that the nodes in $[0,1]$ are uniformly distributed then Y is the position of the boundary node and W is the position of the node immediately to its left.

Let N be the number of phases. We now describe the set of configurations for which $N > 1$. When the boundary node is at $y \in (1-q, 1]$ it will be identified in the first phase iff $w \leqslant 1-q$. Similarly, when $y \in (1-kq, 1-(k-1)q]$, it will be identified in one phase iff $w \leqslant 1-kq$. Therefore the condition for $N > 1$ is $w > 1-kq$ when $y \in (1-kq, 1-(k-1)q]$. These regions are shown in Fig. 7. We now compute the probability of this region, using the joint density. First some notation. Let $L = \max\{k: kq < 1\}$. Let $w_k = 1-(L-(k-1))q$, $1 \leqslant k \leqslant L+1$. Then $w_0 = 0$, $w_L = 1-q$, $w_{L+1} = 1$. Let I_k be the probability of the triangle between w_k and w_{k+1}.

$$I_k = m(m-1)\int_{w_k}^{w_{k+1}} w^{m-2}\,\mathrm{d}w \int_{w}^{w_{k+1}} \mathrm{d}y$$

$$= (w_{k+1}^m - w_k^m) + mw_k^{m-1}(w_k - w_{k+1}).$$

Noting that $w_k - w_{k+1} = q$ we have

$$I_k = (w_{k+1}^m - w_k^m) - mqw_k^{m-1}, \quad 1 \leqslant k \leqslant L,$$

$$I_0 = (w_1^m - w_0^m),$$

$$P(N > 1) = \sum_0^L I_k = \sum_0^L (w_{k+1}^m - w_k^m) - mqw_k^{m-1}$$

$$= 1 - \sum_0^L - mqw_k^{m-1}. \tag{10}$$

Now we need to estimate $q\sum_0^L w_k^{m-1}$. We observe that this is the *lower sum* for the integral $\int_0^1 x^{m-1}\,\mathrm{d}x$ with the partition of $[0,1]$ given by $\pi = \{w_0, w_1, \ldots, w_{L+1}\}$. Since x^{m-1} is a continuous function we know that

$$L(\pi) \leqslant \int_0^1 x^{m-1}\,\mathrm{d}x \leqslant U(\pi),$$

where L and U are lower and upper sums for the integral $\int_0^1 x^{m-1}\,\mathrm{d}x$ with partition π. ($L(\pi) = q\sum_0^L w_k^{m-1}$ and $U(\pi) = q\sum_1^{L+1} w_k^{m-1}$. Also, $U(\pi) - L(\pi) = q(w_{L+1}^{m-1} - w_0^{m-1}) = q(1-0) = q$.) From this we get

$$\int_0^1 x^{m-1}\,\mathrm{d}x - L(\pi) \leqslant (U(\pi) - L(\pi)).$$

We use this in Eq. (10) to obtain

$$P(N > 1) = 1 - m\int_0^1 x^{m-1}\,dx + m\left(\int_0^1 x^{m-1}\,dx - L(\pi)\right).$$

Thus, $P(N > 1) \leqslant m(U(\pi) - L(\pi))$. Noting that $(U(\pi) - L(\pi)) = q$, we have $P(N > 1) \leqslant mq$. Now to find $P(N > 2)$, we need to consider the probability of regions contained inside the shaded triangles. Now observe that we have a self-similar structure. Fig. 8 is just a magnification of one of the triangles in Fig. 7, and inside it those regions for which $N > 2$ are again given by a set of triangles with sides now of length q^2. If we now do a similar computation as for $N > 1$, we obtain $P(N > 2) \leqslant mq^2$. Proceeding thus it easily follows that $P(N > k) \leqslant mq^k$, $\forall k \geqslant 1$. Now $E(N) = \sum_{i=0}^{\infty} P(N > i)$. Let k_1 be an integer such that $mq^{k_1} < 1$. Now we note that $P(N > k_1 + l) \leqslant mq^{k+l} < q^l$. Thus, we have

$$\sum_{i=0}^{\infty} P(N > i) < \sum_{i=0}^{k_1} P(N > i) + \sum_{l=1}^{\infty} q^l$$

$$= \sum_{i=0}^{k_1} P(N > i) + \frac{q}{(1-q)}.$$

Now suppose that $m \to \infty$ and $q(m) \to 0$. Then $E(N) < (2k_1 + 1) + q/(1 - q) = Ck_1$ where $C < 3$. We will take k_1 to be the $\inf\{k > 1: mq^k < 1\}$. With this definition,

$$q^{k_1 - 1} > 1/m,$$

$$(k_1 - 1)\ln q > \ln(1/m),$$

$$(k_1 - 1) < \frac{\ln(1/m)}{\ln q}.$$

Which yields

$$k_1 < 1 + \frac{\ln m}{\ln(1/q)},$$

which finally gives us

$$E(N) < B\frac{\ln m}{\ln(1/q)}. \qquad \square$$

Observe that if q is very small then the number of phases tends to 1. However, each such phase will involve a large amount of sequential search (step 4(b)) to find an interval which contains the boundary node. On the other hand, if q is large, say for instance 1/2, for large values of m, there will be almost no sequential search in any phase, but the number of phases will be large. Therefore, the choice of q determines the speed with which we can identify one boundary node.

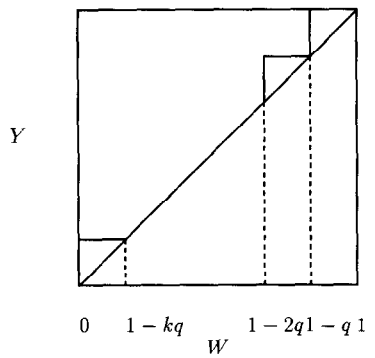

Fig. 7. First execution of step 4(a).

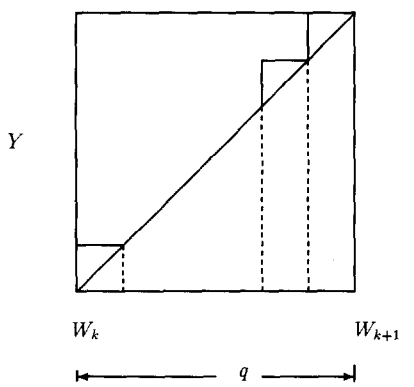

Fig. 8. Recursive structure within each interval.

Theorem 4.2. *If $q = 1/\log_2 m$ then the number of sequential searches in each phase i, where $i < \log m/\log\log m$, is 1 asymptotically as $m \to \infty$.*

Proof. If we assume that the statement of the theorem is true for all phases until $i - 1$ then in phase i we will look for a node in the region $(1 - (\log_2 m)^{-i}, 1]$. The probability that this region will not contain any nodes is given by

$$\left[1 - \frac{1}{(\log_2 m)^i}\right]^m.$$

While $i < \log m/\log\log m$, the above probability takes a value 0 in the limit as $m \to \infty$ since we can write m as $(\log m)^{\log m/\log\log m}$. □

We conducted extensive simulations to study the problem of determining the optimal value of q (i.e., to minimize the number of time steps required to find one

boundary node). We found that for values of m varying from low values like 10 to values as high as 5×10^4, $q = 1/\log_2 m$ yielded the lowest complexity! Infact, the total number of time steps required was $\log m/\log\log m + c$ for some small integer c. We therefore have the following conjecture.

Conjecture 4.3. Let $X_1, X_2, \ldots, X_m$ be i.i.d. uniform $[0, 1]$ r.v.'s. Then,

(i) The expected number of time steps required to find $X_{(m)}$ (the mth order statistic) is bounded below by $\log m/\log\log m + c$ in the limit as $m \to \infty$,

(ii) Algorithm 2 is optimal with $q = 1/\log_2 m$.

In conclusion, we observe that Algorithm 2 is executed iteratively to find each successive boundary node. If $L \ll n$ then the expected time complexity to identify all the boundary nodes is given by (using the conjecture) $L \log m/\log\log m$ where $m = n/L$ (by Corollary 3.4). If L is unrestricted, on the other hand, it takes a constant number of steps to find each boundary node since the number of nodes between succesive boundary nodes is a constant. Therefore, the time complexity in this case is kN_n where k is a constant and N_n is the number of boundary nodes.

5. Summary of results

The results of this paper may be divided into three groups. First, we presented an algorithm for broadcasting that was shown to be optimal. Then utilizing the construction of forbidden regions, we derived approximate expressions for the probability of exactly k boundary nodes in connected configurations.

Next, we studied the asymptotic (as $n \to \infty$) behaviour of N_n, the number of boundary nodes. The results are summarized in Table 1 for the cases when $L(n) \sim n^\alpha$ and when $L(n) \sim \alpha n$. For most of the cases, the number of boundary nodes (or equivalently the number of time steps to broadcast) is L. However, for $L(n) \sim \alpha n$, $\alpha > 3/4$, the value of N_n approaches $3L/4\alpha$.

Finally, we addressed the problem of deciding how nodes would identify themselves as boundary nodes in random connected configurations. We presented an algorithm and discussed the number of time steps required to identify each boundary node. We further conjectured that for $q = 1/\log_2 m$, the algorithm is optimal and finds one boundary node in $\log m/\log\log m + c$ time steps, where m is the number of nodes in an interval of length 1.

6. Conclusions

In this paper we introduced a new system model and studied the problem of broadcasting. In this model nodes communicate via radio rather than telephone lines. As a result *collisions* become an important consideration in any algorithm. We

presented analysis for the expected number of time steps required to broadcast and showed that the algorithm was optimal.

Some obvious generalizations of our basic system model yields several open problems. One generalization of this model is to study systems where the nodes are placed on a 2-dimensional plane. Another generalization of interest is when the nodes are placed on the line as a poisson process. It is conceivable that some of the analysis is easier. A different kind of generalization is if we assume that the transmission radius is not a constant but a random variable. Transmission signals tend to fade as a function of distance from the transmitter. Perhaps a *normal* distribution for the transmission radius would yield interesting results.

It is of interest to study the related problem of *gossiping* in each of these models. We have studied the gossiping problem for the case when nodes are placed on a line $[0, L]$ and our results will be presented in a future paper. In [11] we present a gossiping algorithm that is shown to be optimal asymptotically.

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